Geometric Optimal Control with Applications

Accelerated Graduate Course
Institute of Mathematics for Industry, Kyushu University,

Bernard Bonnard

Inria Sophia Antipolis et Institut de Mathématiques de Bourgogne

9 avenue Savary

e21078 Dijon, France

Monique Chyba

2565 McCarthy the Mall

Department of Mathematics

University of Hawaii

Honolulu, HI 96822, USA

with the help of

Gautier Picot, Aaron Tamura-Sato, Steven Brelsford

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Chapter 1

Introduction and Motivation

This section deals with NMR.

1.1   Introduction

Control theory is the area of application-oriented mathematics that deals with the basic principle underlying the analysis and design of control systems. It is a branch of mathematics that studies the properties of control systems i.e dynamical systems whose behavior can be modified by a command.

Goal: to control an object, to influence its behavior to achieve a desired goal.

Examples: Watts steam engine governor, CD players and automobiles, home temperature controlling system or industrial robots and airplanes autopilots. Control mechanisms are widespread in nature and are used by living organisms to maintain essential variables such as body temperature and blood sugar levels for instance. In engineering, feedback control has a long history: As far back as the early Romans, one finds water levels in aqueducts being kept constant through the use of various combinations of valves. In more moderns days, it is now also found in the medical field as well with the use of optimal control chemotherapy protocols and surgery robotics.

Two Main Lines of Work in Control Theory:

- A good model of the object to be controlled is available and one wants somehow optimize its behavior, this is known as **optimal control**.

- Uncertainty about the model or about the environment in which the object operates. It is **feedback control**.

The best work in systems and control theory makes it clear that the creative modeling of phenomena and the conceptualization of systems using mathematics and the subsequent mathematical analysis of these models to get a deeper understanding of the phenomena are what is central to the field.

1.1.1   Control systems

Differential equations effectively model mathematically the real world from physics such as Newton's law to produce computational model for understanding biological complex process for instance. Let $x \in \mathbb{R}^n$ be the state variable, the evolution of a system whose behavior is determined by its state only is expressed as:

$$\dot{x}(t) = f(x)$$

If an initial value is known x_0, which is equivalent to knowing the state at an initial time t_0: $x_0 = x(t_0)$, the future behavior of the system is completely determined by solving the corresponding Cauchy problem:

$\dot{x}(t) = f(x), x(t_0) = x_0$. The motion of the planets in the solar system are solution of a Cauchy problem, it is a typical example of evolution that cannot be altered. However, as it will be seen later, when analyzing the motion of a spacecraft an external force can be applied by the use of thrusters. This falls into the category of control theory where we can actively influence the evolution of the system. A new parameter, called the control, appear in the differential equation:

$$\dot{x}(t) = f(x(t), u(t))$$

where f is a C^1-mapping, $u(.)$ belongs to an admissible family of functions whose regularity varies depending on the application and that we assume here to be the set of bounded measurable functions and takes its value in a subset of $\mathbb{R}^m$. For control systems, the rate of change $\dot{x}(t)$ and therefore the evolution of the system depends not only on the state itself, but also on the control u that varies in time. The control is chosen depending on the application, the goal might be to steer the system from one state to another, to maximize the terminal value of one of the parameters, or to minimize a certain cost functional for instance.

Controllability

Is it possible to bring the state variable from any initial condition to any final condition in a finite time?

- linear systems
$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t) \\ x(t_0) = x_0 \end{cases}$$

  where $x(t)$ $\mathbb{R}^n$, $u(t)$ $\mathbb{R}^n$, $A(t) \in \mathcal{M}_n(\mathbb{R})$, $B(t) \in \mathcal{M}_{n,m}(\mathbb{R})$ and $r(t) \in \mathcal{M}_{n,1}(\mathbb{R})$ for all $t \in [t_0, t_f]$.
  $\rightarrow$ *Kalman condition*

- nonlinear systems
  $\rightarrow$ *much more complex*
  $\rightarrow$ Poincaré's Reccurence theorem, Poisson-stability, linearization, local controllability

Example

The following example is taken from [**?**]. We assume to have a cart moving horizontally in a frictionless environment. For simplicity, assume that it has unit mass: $m = 1$. We denote its initial position by $x(0)$ and $v(0)$ is its initial velocity. Assuming that no extremal forces are applied, the evolution of the cart is simply given by $x(t) = x(0) + v(0)t$ where t represents the time. If now we assume there is an extremal force pushing on th cart, denoted by the control function $u(.)$, the system becomes $\ddot{x}(t) = u(t)$ which is equivalent to the first oder system:

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = u(t)$$

that can be integrated to obtain

$$x(t) = x(0) + v(0)t + \int_0^t (t-s)u(s)ds, \ \ v(t) = v(0) + \int_0^t u(s)ds.$$

Assume that the force satisfies the constraint $|u(t)| \leq 1$, and consider the problem of steering the system to the origin with zero speed. Assuming $(x(0), v(0)) = (2, 2)$, this goal is achieved by the open-loop control (verify!)

$$u(t) = \begin{cases} -1 & \text{if} \quad 0 \leq t < 4, \\ 1 & \text{if} \quad 4 \leq t < 6, \\ 0 & \text{if} \quad \ \ 6 \leq t. \end{cases}$$

Note that because of uniqueness of solutions, this control would not accomplish the same task in connection with any other initial data different from $(2, 2)$.

Stabilization

How can we make a control system insensitive to perturbations?
Example: if (x_e, u_e) is an equilibrium point of the autonomous control system

$$\dot{x}(t) = f(x(t), u(t))$$

i.e

$$f(x_e, u_e) = 0.$$

Does it exist a control u such that, for all $\epsilon > 0$, there exists $\eta > 0$ such that, for all $x_0 \in B(x_e, \eta)$ and all $t \geq 0$, the solution to the system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ \quad\quad x(t_0) = x_0 \end{cases}$$

satisfies $||x(t) - x_e|| \leq \epsilon$?

- linear systems $\rightarrow$ controlability

- nonlinear systems $\rightarrow$ Lyapunov functions

See [**?**] for asymptotic stabilization question regarding the cart example above.

1.1.2   Optimal Control

Given a control system, can we determine the optimal solutions for a given optimization criterion? When the initial and final states are fixed, it is equivalent to find the solution to the following boundary value problem:

$$\begin{cases} \quad\quad\quad \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \in M_0, \ x(t_f) = x_f \in M_1 \end{cases}$$

which *minimizes* the cost

$$\min_{u(.)\in U} \int_{t_0}^{t_f} f_0(t, x(t), u(t))dt + g(t_f, x_f)$$

where $f_0 : \mathbf{R} \times M \times U \to \mathbf{R}$ is smooth and $g : \mathbf{R} \times M \to \mathbf{R}$ is continuous.

- $\int_{t_0}^{t_f} f_0(t, x(t), u(t))dt$: *Lagrange* cost

- $g(t_f, x_f)$: *Mayer* cost

In 1696, Johan Bernoulli challenged his colleagues with the following problem: Consider two points A and B such that A is above B. Assume that a object is located a the point A with no initial velocity and is only subjected to the gravity. What is the curve between A and B so that the object travels from A to B in minimal time ? We know that the straight line is the shortest way between two points. Is it the fastest way? NO! The fastest way is a cycloid arc.
Optimal control theory is at the crossroad of:

1. Theory of Differential Equations/Dynamical Systems (Finding solutions to differential systems, problem of existence and unicity of optimal solutions)

2. Differential geometry (optimal synthesis strongly depends on the geometric properties of the problem, modern theory of optimal control)

3. Optimization

4. Modeling (relevance of the way that an optimal control problem is set up)

5. Numerical Analysis (numerical methods to approximate optimal solutions)
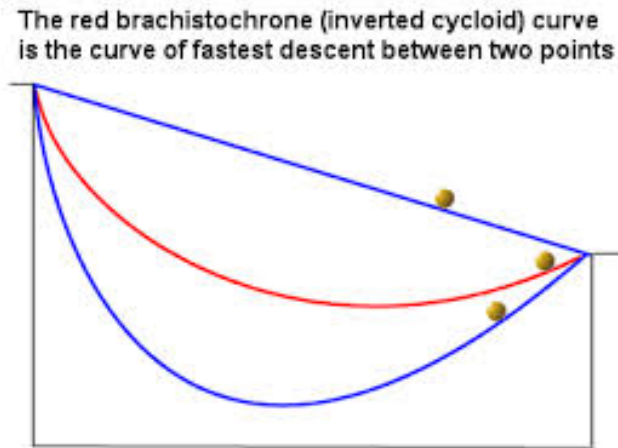
6. Applications (solving real world problems)

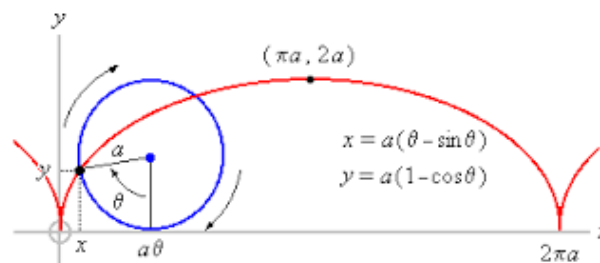Figure 1.1: Cycloid is the solution to the Brachistochrone problem.

Figure 1.2: Cycloid's equations.

1.2   Examples of Applications

1.2.1   Identification of the Fragmentation Role in the Amyloid Assembling Processes and Application to their Optimization

Goal: Apply techniques from geometric control to a kinetic model of amyloid formation which will take into account the contribution of fragmentation to the de novo creation of templating interfaces to design optimal strategies for accelerating the current amplification protocols, such as the Protein Misfolding Cyclic Amplification (PMCA). The objective is to reduce the time needed to diagnose many neurodegenerative diseases. Expected results could be generalized to all neuropathologies caused by protein misassembly, such as Prion, Alzheimer's and Parkinson's diseases, which involve amyloid fibril formation.

Fibril fragmentation

Fibril fragmentation has been reported to enhance the polymerization process underlying the behavior of some specific prions. The mechanism of this enhancement is mainly based on the generation of supernuclei [18], which according to the Oosawa general model [24, 25] create a shortcut in polymerization. It has been proposed that the fragmentation process was at the origin of replication and propagation of pathogenic structural information of prions in general. According to this theory (which is largely accepted in the prion field and is being extended to other pathologies involving amyloid formations) a certain perturbation leads to a structural change in the native protein (unable to form an amyloid) thus creating a conformer prone to form amyloid assemblies. Once amyloid assemblies form, they serve as templates and convert the native protein into amyloidogenic in an apparently autocatalytic process. Nonetheless, a number of

unanswered questions remain concerning the experimental evidence of such an autocatalytic propagation. There are two physicochemical phenomena in the amyloidogenic process which cannot be ignored: the first one is the structural switch which triggers the formation of the first assembly; the second phenomenon is the fragmentation based amplification of the amyloidogenic process by the de novo generation of the templating interfaces. The former phenomenon has been extensively explored, its molecular mechanisms are well understood, and several mathematical models have been developed. However, there is a significant lack of knowledge concerning the fragmentation process and the de novo generation of templating interfaces, both in the mechanisms of its occurrence and its contribution to the acceleration of the pathology.

Compartmental model of amyloid formation

A first attempt of optimizing the PMCA by using a mathematical model has been introduced in [11] and analyzed in [6, 8]. This prior model takes into account only the size dependence of the polymers, assuming that two polymers with the same length have exactly the same behavior. It also assumes that the monomers saturate the substrate, so that the polymerization and fragmentation intensities depend only on a control parameter which represents the action of the experimentalists on the dynamics of the system (by means of *sonication* in the case of the PMCA, see [11]).

We denote by $x_i^l(t)$, $l = 1, \cdots, k_i$, the density of polymers of size i in compartment l at a given time t. The corresponding rate of change due to elongation is then described as follows:

$$r(u(t))\Big[\sum_{s=1}^{k_{i-1}} \tau_{i-1}^{l,s} x_{i-1}^s - \sum_{r=1}^{k_{i+1}} \tau_i^{r,l} x_i^l \Big].$$

The parameter $\tau_{i-1}^{l,s}$ is the growth rate of polymers of size $i-1$ in compartment s that grow in compartment l of polymers of size i, and $\tau_i^{r,l}$ is the growth rate of polymers of size i in compartment l that grow into polymers of size $i+1$ (in compartment r). The control $u(t)$ stands for the intensity of the sonicator. The fragmentation rate of change is expressed by the fact that polymers of a given size and given compartment fragment into polymers of a given size and compartment at different rates. More precisely, we have:

$$u(t)\Big[2 \sum_{j=i+1}^{n} \sum_{s=1}^{k_j} \beta_j^s \kappa_{ij}^{l,s} x_j^s - \beta_i^l x_i^l \Big]$$

where β_i^l (β_i^s) represents the fragmentation coefficient of polymer of size i in compartment l (s) and the coefficient $\kappa_{ij}^{l,s}$ captures the fraction of polymer of size j that fragment from compartment s into size i polymer in compartment l. An illustration of some possible interactions between different compartments is shown in Figure 1.3. The experiments will determine the fragmentation rules between the compartments which will translate into relations among the parameters $\kappa_{ij}^{l,s}$. It is expected that there is a limitation on the polymers which can be fragmented into a given polymer, and therefore many of the $\kappa_{ij}^{l,s}$ will actually be zero. This is important to allow for an analysis of the system. It is also possible that the standard assumptions on $\kappa_{ij}^{r,s}$ will still hold:

$$\sum_{i=1}^{j-1} \sum_{r=1}^{k_i} \kappa_{ij}^{r,s} = 1, \quad \sum_{i=1}^{j-1} i \sum_{r=1}^{k_i} \kappa_{ij}^{r,s} = \frac{j}{2}.$$

To summarize, we propose a model of the form:

$$\dot{x}_i^l(t) = r(u(t))\Big[\sum_{s=1}^{k_{i-1}} \tau_{i-1}^{l,s} x_{i-1}^s(t) - \sum_{r=1}^{k_{i+1}} \tau_i^{r,l} x_i^l(t) \Big] + u(t)\Big[2 \sum_{j=i+1}^{n} \sum_{s=1}^{k_j} \beta_j^s \kappa_{ij}^{l,s} x_j^s(t) - \beta_i^l x_i^l(t) \Big]. \qquad (1.1)$$

As in [8], equation (1.1) can be written in a matrix form:

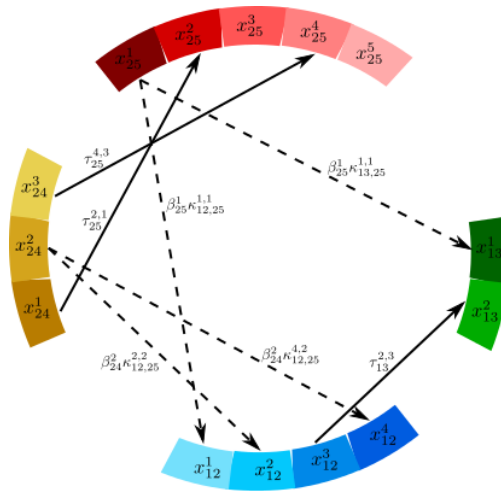$$\dot{x}(t) = \big(u(t)A + r(u(t))B\big)x(t).$$

Figure 1.3: Possible interaction between different compartments of polymers of sizes $12, 13, 24, 25$. Solid arrows indicate growth, dashed arrows indicate fragmentation.

However, the growth matrix, B, and the fragmentation matrix, A, as well as vector $x(t)$, now have a block structure with blocks corresponding to the different compartments. More specifically, we have

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \; A = \begin{pmatrix} 0 & & & \\ & -A_1 & & (2K_{ij}) \\ & & \ddots & \\ 0 & & & \\ & & & -A_n \end{pmatrix}, \; B = \begin{pmatrix} -D_1 & & & \\ T_1 & -D_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & T_{n-2} & -D_{n-1} \\ & & & T_{n-1} & 0 \end{pmatrix},$$

where $x_i(t) = (x_i^1(t), \dots, x_i^{k_i}(t))^\top$, $T_i = (\tau_i^{r,s})$ is a $k_{i+1} \times k_i$ matrix representing growth rates of polymers of size i, $D_i = \mathrm{diag}\left(\sum_{r=1}^{k_{i+1}} \tau_i^{r,s} \right)_{s=1}^{k_i}$, $A_i = \mathrm{diag}(\beta_i^s)_{s=1}^{k_i}$, and $K_{ij} = (\beta_j^s \kappa_{ij}^{r,s})$ is a $k_i \times k_j$ fragmentation matrix for polymers of size j.

If matricies A and B do not depend on x and t, then model (1.1) is a linear system with a constant matrix which is irreducible and has non-negative off-diagonal entries. Hence, we can again pursue a model analysis based on the dominant Perron eigenvalue. It is then possible to show that there is an optimal value of the control, u^*, for which the Perron eigenvalue, $\lambda_P(u)$ attains its global maximum: Assume that $r : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exist $q > 0$ and $r_q \geq 0$ such that

$$r(u) = r_0 + r_q u^{-q} + o(u^{-q}), \text{ as } u \to +\infty$$

Then there exists u^* such that $\lambda_P(u) \leq \lambda_P(u^*) \; \forall u \geq 0$. However, the growth rate matrix, B, may, in general, depend on time and/or polymer concentration, i.e. $B = B(t, x(t))$, thus making the above approach inapplicable. In such a case, we may need to employ to extensive computer simulations to obtain a better understanding about the system behavior.

- The parameters will be determined experimentally. However, the complexity of the system and large number of parameters will require implementation of parameter estimation methodologies to help guide the experimentalists (standard Markov Chain Monte Carlo (MCMC) techniques).

- Once the compartments and the communication between them is well identified, the behavior inside each compartment also needs to be determined. In particular the in vitro elongation of the fibrils appears to saturate after some time and the polymerization process is then blocked. This saturation effect has to be understood and included in the model. After adding the saturation effect the model becomes nonlinear, which raises new challenging mathematical questions. The preliminary work on the linear version in [5, 6, 8] is a solid base on which we can rely to address this new nonlinear problem.

Optimization of amplification protocols

Since incubation of a disease triggered by prions can take place over very long period of time, an important question is the optimization of the templating, elongation, and polymerization processes to accelerate the detection of the protein in an affected person. It is particularly relevant to PMCA, which is a technique to simulate an accelerated replication process for prions in a laboratory environment [28] by creating a cyclic scheme that alternates incubation phases to allow lengthening of the abnormal prion with sonication phases to break the polymers into smaller ones. Typically, [27], during the PMCA the incubation phase (no soni-cation) is more than 30 times the duration of the sonication phase (at a constant frequency) and alteration of these two phases takes place over 48 hours. This correspond to a bang-bang strategy with the control (sonication intensity) switching a finite number of times between its minimum and maximum values. Since the introduction of the PMCA scientists have tried to improve the protocol in various way [9, 13, 23], but there is no literature on the role that singular arcs could play to design a more efficient sonication scheme beside a first approach with the non compartmental model [6, 8].

The general expression for our system is of the form:

$$\dot{x}(t) = [Au(t) + B(t, x(t))r(u(t))]x(t), \tag{1.2}$$

$$x(0) = x_0 > 0, \tag{1.3}$$

where $x = (x_1^1, \cdots, x_1^{k_1}, \cdots, x_n^1, \cdots, x_n^{k_n}) \in R^m$, $m = \sum_{i=1}^{n} k_i$. Notice that the matrix A is constant since we assume that the fragmentation coefficients stay constant throughout the protocol. However, the elongation coefficients might vary with time to reflect the saturation hypothesis. This implies that the matrix B is not constant but can depend explicitly on t or on the current density of polymers $x(t)$.

Optimal Cost: final density of polymers , $c(x(T)) = \sum_{i=1}^{n}(i \sum_{j=1}^{k_i} x_i^j(T))$.

As in [8], we make here the assumption that the function r in (1.1) is a decreasing convex function. This will be checked experimentally, and adapted if it is necessary in further work. Using a reparametrization and some assumption on r, we can rewrite the optimal problem as an affine single-input system:

$$\dot{x}(t) = f_0(t, x(t)) + f_1(t, x(t))u(t), \tag{1.4}$$

$$x(0) = x_0 > 0, \tag{1.5}$$

$$\min_{u_{\min} \leq u \leq u_{\max}} -\psi x(T), \tag{1.6}$$

where $f_0(t, x(t)) = B(t, x(t))$ and $f_1(t, x(t)) = Ax(t) + aB(t, x(t))$, $a < 0$. Our optimal control problem is in Mayer form with fixed time T but not constraints on the terminal state $x(T)$.

1.2.2   Orbital Transfer - Rendezvous Mission to Mini-Moons

Goal: use geometric optimal control theory to compute time-minimizing and energy-minimizing space trans-fers in the Earth-Moon system.

The planar restricted 3-body problem

The first step is to model the motion of a spacecraft in the Earth-Moon system. Given that the eccentricity of the Moon's orbit around the Earth is small (≈ 0.05) and that the spacecraft has, obviously, no influence on the motion of the two planets, the dynamics of the spacecraft can be model by the planar restricted 3 body problem. In [?], the author defines the problem as follows: "Two bodies revole around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of the three bodies is to describe the motion of the third body". The two revolving bodies are commomly called *the primaries* and can be considered two point masses M_1 and M_2. To derive the simplest form of the equations of motion of the third body, we assume that $M_1 + M_2 = 1$ and that the distance unit is the distance between M_1 and M_2. As a result, the angular velocity of the two primaries is normalized to 1. Instead of using a fixed frame $\{G, X, Y\}$, we can use a dynamic rotating frame $\{G, x, y\}$
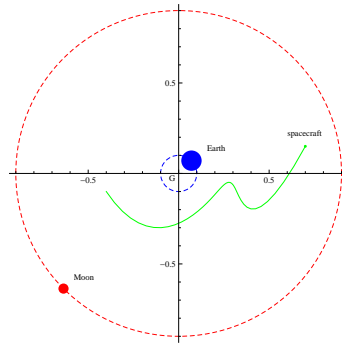
Figure 1.4: The Earth-Moon-spacecraft system seen as the circular restricted 3-body problem. The blue dashed line is the orbit of the Earth and the red one is the orbit of the Moon. The trajectory of spacecraft lies in the plan defined by the circular motion of the Earth and the Moon.
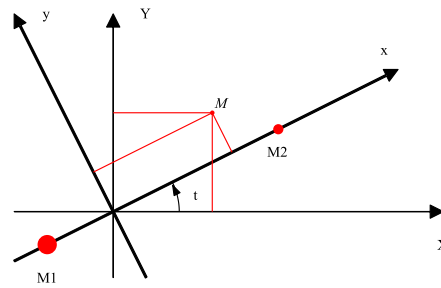

Figure 1.5: Comparison between the fixed coordinates $\{X, Y\}$ and the rotating coordinates $\{x, y\}$ at time t.

which rotates with the same angular velocity as the Earth and the Moon. Denoting t the time variable, the rotating frame is simply obtained by applying a rotation of angle t to the fixed coordinates (X, Y) (the angular velocity being constant equal to 1). Therefore, the relation betwen the two systems of coordinates writes

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(t)x + \sin(t)y \\ -\sin(t)x + \cos(t)y \end{pmatrix}.$$

Denote $\mu = \frac{M_2}{M_1 + M_2}$ the mass ratio. In the rotating frame, the Earth and the Moon have respective masses $1 - \mu$ and μ and are respectively located at the fixed points $(-\mu, 0)$ and $(1 - \mu, 0)$. The equations of motion of the the spacecraft are expressed in thevery simple form

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} \end{cases}$$

where $V(x, y) = \frac{1-\mu}{\varrho_1^3} + \frac{\mu}{\varrho_2^3}$ is the mechanical potential of the system, $\varrho_1 = \sqrt{(x+\mu)^2 + y^2}$ is the distance between the spacecraft and the Earth, and $\varrho_2 = \sqrt{(x-1+\mu)^2 + y^2}$ is the distance between the spacecraft and the Moon. There are 5 possible regions of motion, know as the *Hill regions*. These regions are defined by the value of the total energy of the system $E(x, y, \dot{x}, \dot{y}) = \frac{\dot{x}^2 = \dot{y}^2}{2} - V(x, y)$ which is a first integral of the problem (the total energy is constant along a trajectory solution to the problem.) Actually, the totoplogy of the Hill regions is determined with respect to the total energy at the equilibrium points of the system. The equilibrium points are the critical points of the mechanical potential, solution to the equation $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$. There are two categories of equilibrium points:
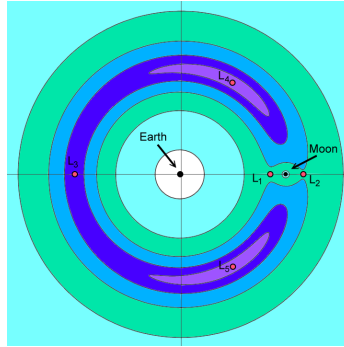
Figure 1.6: The Hill regions of the planar restricted 3-body problem

- the Euler points L_1, L_2, L_3 which are three colinear points located on the axis $y = 0$, with $x_1 \simeq 1.1557$, $x_2 \simeq 0.8369$, $x_1 \simeq -1.0051$.

- the Lagrange points L_4, L_5 which form equilateral triangles with the primaries.
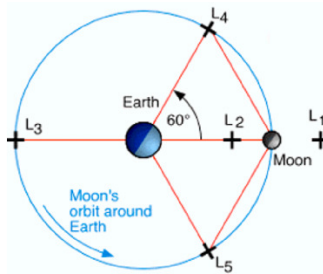

Figure 1.7: Equilibrium points of the Earth-Moon

The controlled restricted 3-Body problem.

The motion of the spacecraft in the Earth-Moon system is not fully determined by the motion of the two planets. Indded, the thrust provided by the engines of the spacecraft can be used to alter the accelaration of the spacecraft. In other words, it is the control that we have on the spacecraft dynamics. The influence of this control must be added to the equations of motion **??**. Thus, we get the equations of the controlled restricted 3-body problem

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x} + u_1 \\ \ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y} + u_2. \end{cases}$$

where $u = (u_1, u_2)$ is the control term which represents the impact of the engine thrust on the spacecraft motion. We introduce the coordinates $q = (x, y, \dot{x}, \dot{y})$. As a result, the controlled equations of motion are written as the bi-input control system

$$\dot{q} = F_0(q) + F_1(q)u_1 + F_2(q)u_2$$

where

$$F_0(q) = \begin{pmatrix} q_3 \\ q_4 \\ 2q_4 + q_1 - (1-\mu)\frac{q_1+\mu}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_1-1+\mu}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \\ -2q_3 + q_2 - (1-\mu)\frac{q_2}{((q_1+\mu)^2+q_2^2)^{\frac{3}{2}}} - \mu\frac{q_2}{((q_1-1+\mu)^2+q_2^2)^{\frac{3}{2}}} \end{pmatrix},$$

$$F_1(q) = \frac{\partial}{\partial q_3}, \; F_2(q) = \frac{\partial}{\partial q_4}.$$

Time-minimal problem

Our objective is to minimize the transfer time between the geostationary orbit $\mathcal{O}_G$ and a circular parking orbit $\mathcal{O}_L$ around the Moon when low-thrust is applied. This problem consists of solving the optimal control problem

$$\begin{cases} \dot{q} = F_0(q) + \epsilon(F_1(q)u_1 + F_2(q)u_2), \epsilon > 0 \\[2mm] \min_{u(.) \in B_{\mathbb{R}^2}(0,1)} \int_{t_0}^{t_f} dt \\[2mm] q(0) \in \mathcal{O}_G, \; q(t_f) \in \mathcal{O}_L. \end{cases}$$

In other words, we want to find the solutions to the controlled equations of motion of the spacecraft, starting from the geostationary orbit and ending at the circular parking orbit around the Moon, which minmize the transfer time $\int_{t_0}^{t_f}$. The control bound ϵ represents maximum thrust allowed, which can be arbitarily set to any sufficiently small value. Unfortunately, this optimal control problem problem can not be solved analytically. It is highly non-linear and has singularities. However, necessary optimality conditions and numerical calculations can produce simulatations of locally optimal solutions. The methodoly is the following

1. Apply Pontryagin's Maximum Principle which provides necessary optimality conditions¿

2. Turn the Pseudo-Hamiltonian system deduced from the Application of the Pontryagin's Maximum Principle into a true Hamiltonian system. This can be done by using the Implicit function Theorem.

3. Use shooting method to compute candidate solutions of the problem.

4. Check the local-optimality of these extremals by using second order optimality condition. To do so, we need to compute the first conjugate time along each extremal and verify that it is greater than the transfer time.

Acknowledgments

Bibliography

[1] A. Agrachev, A. Sarychev, *Abnormal sub-Reimannian geodesics: Morse index and rigidity*, Ann. Inst. H. Poincaré Anal. Non Linéaire 13, No. 6 (1996), pp.635-690.

[2] M. Bonnard and M. Chyba, *Singular trajectories and their role in control theory*, Springer-Verlag, Berlin, 2003.

[3] G.A. Bliss, *Lectures on the Calculus of Variations*, Univ. of Chicago Press, Chicago, 1946.

[4] B. Bonnard, I. Kupka, *Théorie des singularités de l'application entrée-sortie et optimalité des trajectoires singuliéres*, Forum Math. 5, No. 2 (1992), pp.111-159.

[5] V. Calvez, P. Gabriel, and S. Gaubert, *Non-linear eigenvalue problems arising from growth maximization of positive linear dynamical systems*, Proceedings of IEEE 53rd Annual Conference on Decision and Control (CDC 2014), Los Angeles, pp. 1600–1607.

[6] M. Chyba, J.-M. Coron, P. Gabriel, A. Jacquemard, G. Patterson, G. Picot, and P. Shang, *Optimal geometric control applied to the protein misfolding cyclic amplification process*, Acta Appl. Math., in press (2015), DOI 10.1007/s10440-014-9950-8.

[7] M. Chyba, G. Patterson, G. Picot, *et al.*, *Designing rendezvous missions with mini-moons using geometric optimal control*, J. Ind. Manag. Optim., 10 (2014), pp. 477–501.

[8] J.-M. Coron, P. Gabriel, and P. Shang, *Optimization of an amplification protocol for misfolded proteins by using relaxed control*, J. Math. Biol., 70 (2015), pp. 289–327.

[9] N. Fernández-Borges and J. Castilla, *PMCA. A decade of in vitro prion replication*, Current Chem. Biol., 4(2010), pp. 200–207.

[10] R. Gabasov, F.M. Kirilova, *High order necessary conditions for optimality*, SIAM J Control Optim. 10, No. 1 (1972), pp.127-168.

[11] P. Gabriel, *Équations de transport-fragmentation et applications aux maladies à prions*, PhD thesis, Université Pierre et Marie Curie-Paris VI, 2011.

[12] C. Godbillon, *Geométrie différentielle et mécanique analytic*, Hermann, Paris, 1969.

[13] N. Gonzalez-Montalban, N. Makarava, V. G. Ostapchenko, *et al.*, *Highly efficient protein misfolding cyclic amplification*, PLoS Pathog., 7(2009), e1001277.

[14] D. Henrion, J. B. Lasserre, and J. Lofberg, *Gloptipoly 3: Moments, optimization and semidefinite programming*, Optim. Methods Software, 4 (2009), pp. 761–779.

[15] H. Hermes, *Lie algebras of vector fields and local approximation of attainable sets*, SIAM J Control Optim. 16, No. 5 (1978), pp.715-727.

[16] R.M. Hirschorn, *Controllability in nonlinear systems*, J Differential Equations 19, No. 1 (1975), pp.46-61.

[17] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, 52 (1997), Cambridge University Press, Cambridge.

[18] V. Jurdjevic, I. Kupka, *Control systems subordinated to a group action*, J Differential Equations 39, No. 2 (1981), pp.186-211.

[19] D. Kashchiev, *Nucleation*, Butterworth-Heinemann, 2000.

[20] T. P.J. Knowles, C. A. Waudby, G. L. Devlin, S. I.A. Cohen, A. Aguzzi, M. Vendruscolo, E. M. Terentjev, M. E. Welland, and C. M. Dobson, *An analytical solution to the kinetics of breakable filament assembly*, Science, 326 (2009), pp. 1533–1537.

[21] A.J. Krener, *The high order maximum principle and its application to singular extremals*, SIAM J Control Optim. 15, No. 2 (1977), pp.256-293.

[22] C. Lobry, *Controllability of non-linear systems on compact manifolds*, SIAM J. Control and Optimization 1 (1974), pp.1-4.

[23] J. Masel, V. A.A. Jansen, and M. A. Nowak, *Quantifying the kinetic parameters of prion replication*, Biophys. Chem., 77 (1999), pp. 139–152.

[24] C. E. Mays, W. Titlow, T. Seward, *et al.*, *Enhancement of protein misfolding cyclic amplification by using concentrated cellular prion protein source*, Biochem. Biophys. Research Comm., 388 (2009), pp. 306–310.

[25] F. Oosawa and S. Asakura, *Thermodynamics of the Polymerization of Protein*, Academic Press, 1975.

[26] F. Oosawa and M. Kasai, *A theory of linear and helical aggregations of macromolecules*, J. Mol. Biol., 4 (1962), pp. 10–21.

[27] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, *et al.*, *The Mathematical Theory of Optimal Processes*, John Wiley & Sons, New York, 1962.

[28] P. Saa and J. Castilla and C. Soto, *Ultra-efficient Replication of Infectious Prions by Automated Protein Misfolding Cyclic Amplification*, Biological Chemistry 281(16)(2006), pp. 35245- 3525.

[29] G. P. Saborio, B. Permanne, and C. Soto, *Sensitive detection of pathological prion protein by cyclic amplification of protein misfolding*, Nature, 411(2001), pp. 810–813.

[30] H.J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Am. Math. Soc. 180 (1973), pp.171-188.

[31] H.J. Sussmann, V. Jurdjevic, *Controllability of non-linear systems*, J Differential Equations 12 (1972), pp.95-116.

[32] W.-F. Xue, S. W. Homans, and S. E. Radford, *Systematic analysis of nucleation-dependent polymerization reveals new insights into the mechanism of amyloid self-assembly*, Proc. Natl. Acad. Sci., 105 (2008), pp. 8926–8931.